Beurling zeta functions with infinitely many zeros on a prescribed contour

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Abstract

One of the profound insights from Riemann's 1859 paper is that the zeros of the Riemann zeta function ζ have an "influence" on the distribution of the prime numbers. In this context, Ingham asked the following question. Suppose one knows that ζ has no zeros in a certain region $\Re s > 1 - \eta(\Im s)$ for a certain function $0 < \eta < 1/2$. What bound can one then obtain for the error term in the prime number theorem? This question was answered more or less definitively by Pintz in 1980.

The same question can be studied in the context of *Beurling generalized number systems*, where one considers sequences of *generalized primes* and *generalized integers*, and their associated *Beurling zeta function*. Recently, Révész established the analogon of Pintz' theorem in this setting.

In this talk, I will discuss the following result: given any "nice" function η, there exists a Beurling number system whose zeta function has infinitely many zeros on the contour $\Re s = 1 - \eta(\Im s)$ and none to the right, and for which the error term in the prime number theorem matches in size with the conclusion of the Pintz–Révész theorem, hereby showing that this theorem is sharp.