


BELGIAN MATHEMATICAL SOCIETY

Comité National de Mathématique CNM

C W N M

NCW Nationaal Comite voor Wiskunde

BMS-NCM NEWS: the Newsletter of the Belgian Mathematical Society and the National Committee for Mathematics

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BMS-NCM NEWS

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No 79, September 15, 2010

Letter from the editor

Welcome
to our "September 15, 2010-Newsletter"
Have nice semester!

Françoise

Contents

1 News from the BMS

2010 PhD-Day

On September 13, the Belgian Mathematical Society organized its third **PhD-Day**.

This meeting was a great success!
Thank you so much to each of the participants! (sponsor, PhD student and other.)!

Our next PhD-Day is going to take place in 2012.

It is also a pleasure and honor to communicate the . . .

Winners for the "best posters" :

- 1st prize, 50 EUR voucher for books at EMS Publishing house:

  Hilde De Ridder (UGent): *The Cauchy-Kovalevska Extension Theorem in Discrete Clifford Analysis.*

- 2nd prize, book 'Geometry Revealed' by M. Berger, Springer Verlag:

  Dennis Dreesen (KULAK): *Hilbert space compression for groups equipped with proper length functions.*

- ex-aequo third prizes, books 'Wiskunde + Afrika' from VUB Press and a beautiful wooden double helix from Arabesk:
  - Bertrand Desmons (UMons): One-dimensional wrinkling of thin membranes.
  - Sarah De Wachter (VUB): Quantifying domains using approach spaces.
  - Kim Vercammen (KULAK): Complete LR-structures on solvable Lie algebras.

You will find the posters at the end of this Newsletter.

But . . . since sizes of files had to be reduced for sending email (so quality is sometimes not as good as it should be), a special web page is dedicated to them:

http://bms.ulb.ac.be/phdday/winningposters10.php

Emails of the winners: hdr@cage.ugent.be, Dennis.Dreesen@kuleuven-kortrijk.be, Bertrand.Desmons@umons.ac.be, sdewacht@vub.ac.be, Kim.Vercammen@kuleuven-kortrijk.be

During this whole day, participants had the opportunity to visit the stand of *Arabesk*, one of the sponsors. But one can also visit the webpage, enjoy and get some of their products at any moment!

Arabesk

ARABESK sells products whereby artists and designers have been inspired by physics, mathematics and logic; the results are surprising puzzles, games and objects, sometimes complex, sometimes simple, but always intriguing and always beautiful in their appearances.

ARE YOU CURIOUS??? You can have a look at the online catalog of Arabesk at the address http//www.arabesk.nl

Addresses of *Arabesk*: Avenue Concordia 17-B, 3062 LA Rotterdam
+31 10-2140361, +31 6-51-472492, http//www.arabesk.nl, arabesk@arabesk.nl

2 Meetings, Conferences, Lectures

2.1 UMons

Les services

d'Analyse Mathématique et de Probabilités et Statistique

organisent une journée de rencontres et d'exposés dans le cadre de l'EDT Mathématique:

Analyse fonctionnelle, 16 novembre 2010, UMons

Les deux conférenciers sont:

- 10h30 Etienne Matheron (Université de Lens): *Topologie, théorie ergodique et structure des ensembles entiers.*
- 14h30 Jean Saint-Raymond (Université de Paris 6). Thème: théorie descriptive des ensembles et applications à l'analyse fonctionnelle.

La réunion aura lieu au bâtiment "le Pentagone" (local 0A11), avenue du champ de Mars, Mons.
Informations and contacts: catherine.finet@umons.ac.be , kg.grosse-erdmann@umons.ac.be

3 PhD theses

Coring techniques and monoidal categories applied to Hopf algebras and their generalizations
Kris Janssen

June 7, 2010, VUB

Supervisor: Stefaan Caenepeel

Summary

In the past decades many generalizations of Hopf algebras have appeared in the literature. In this work we are particularly interested in multiplier Hopf algebras (MHAs) and Hopf group coalgebras, especially in their categorical behavior.

In the second chapter we develop a theory of *group corings*, notably a Galois (descent) theory for such objects. As an application of the latter we obtain a Structure Theorem for relative Hopf group modules over

a faithfully flat Galois extension. We also discuss the relation between group corings and the dual notion of group-graded ring, and introduce and study *strong* group corings, dualizing strongly group-graded rings.

In the third chapter we present two approaches to better understand MHAs from a categorical point of view. The first one makes use of the notion of a *multiplier bialgebra*, so that a MHA is a multiplier bialgebra along with some kind of antipode, as it is the case classically. The other one consists in the development of a general theory of so-called *Kleisli-Hopf algebras*. Examples of these are provided by a broad class of MHAs, as well as by Hopf group coalgebras.

In the final chapter we generalize usual (co)actions of a Hopf algebra on an algebra to *partial* (co)actions, making use of coring techniques. Several examples are given, of which partial group actions are the most basic and motivational.

The first chapter captures some (well-known) generalities on monoidal categories, Hopf algebras and corings.

Surfaces in three-dimensional Euclidean and Minkowski space, in particular a study of Weingarten surfaces
Wendy Goemans

3 September 2010, KUL

Promotor and co-promotor: F. Dillen Copromotor: I. Van de Woestyne

Summary

In this thesis situated in the area of differential geometry, surfaces and hypersurfaces are studied that can be generated from curves. More precisely, so-called translation surfaces, tensor product surfaces, and translation hypersurfaces are the main subjects under investigation.

Firstly, a translation surface arises when a curve is translated over another curve. The ambient space of the translation surfaces under consideration is either the Euclidean or the Minkowski 3-space. For these surfaces, several curvature conditions concerning the Gaussian curvature, the mean curvature, the second Gaussian curvature and the second mean curvature are examined. This has led to theorems characterizing constant curvature translation surfaces and full classification theorems of translation surfaces exhibiting a functional relation between the Gaussian curvature and the mean curvature. These surfaces are called Weingarten surfaces.

Secondly, by taking the tensor product of two curves, a tensor product surface is obtained. Here, we consider semi-Euclidean spaces of arbitrary dimension and index as ambient spaces for the two curves. A full classification of minimal tensor product surfaces has been realized.

Thirdly, translation surfaces can be generalized naturally to translation hypersurfaces. However, while the translation surfaces and tensor product surfaces treated previously are assumed to be non-degenerate, the translation hypersurfaces under consideration are lightlike. It is shown that every lightlike translation hypersurface must be part of a hyperplane.

The electronic version of this dissertation is publicly available and can be reached by browsing the catalogue of the university library at http://bib.kuleuven.be/index.php

4 Miscellaneous

4.1 Fields Medal and other Prizes

International Mathematical Union (IMU) announced the 2010 Fields Medal winners at the opening of the International Congress of Mathematicians (ICM 2010) in Hyderabad, India.

2010 Fields Medal Winners

- *Elon Lindenstrauss*

  ICM says that Elon Lindenstrauss from Princeton University is being awarded the 2010 Fields Medal "for his results on measure rigidity in ergodic theory, and their applications to number theory".

- *Ngô Bao Châu*

  receives the 2010 Fields Medal "for his proof of the Fundamental Lemma in the theory of automorphic forms through the introduction of new algebro-geometric methods". He is currently a Professor in the Faculté des Sciences at Orsay and Member of the Institute for Advanced Study in Princeton. In September 2010, he will be starting at the University of Chicago.

- *Stanislav Smirnov*

  receives the 2010 Fields medal for the "proof of conformal invariance of percolation and the planar Ising model in statistical physics". He is a professor at University of Geneva, Switzerland.

- *Cédric Villani*

  from Institut Henri Poincaré (IHP) in Paris receives the 2010 Fields Medal "for his proofs of nonlinear Landau damping and convergence to equilibrium for the Boltzmann equation".

In addition to the Fields Medals, the winners of three other top Math awards, Nevanlinna Prize, Gauss Prize, and a new Chern Prize, were also announced at ICM 2010.

Nevanlinna Prize

The Nevanlinna Prize is being awarded since 1982 to young scientists who have done outstanding research in theoretical computer science. This prize is established to honor the Finnish mathematician Rolf Nevanlinna. Nevanlinna Prize is is awarded once every 4 years at the ICM meeting.

Daniel Spielman,

from Yale University, is the the winner of Nevanlinna Prize 2010 for his contributions to"Linear Programming, algorithms for graph-based codes and applications of graph theory to Numerical Computing".

Gauss Prize

Gauss Prize is being awarded since 2006 to the top mathematicians in the field of applied mathematics, giving importance to the mathematical results that have spawned new areas of practical applications.

Yves Meyer,

Professor Emeritus at École Normale Supérieure de Cachan, France, is the winner of Gauss Prize 2010 for his fundamental contributions to number theory, operator theory and harmonic analysis, and his pivotal role in the development of wavelets theory, so used nowadays in signal analysis and statistics.

Chern Prize

IMU announced a new award called Cher Prize this year for recognizing lifetime achievement of mathematicians. Chern prize is established to honor the late Chinese mathematician Shiing-Shen Chern.

Louis Nirenberg,

from Courant Institute of Mathematical Sciences, New York University (NYU), is the first winner of the Chen Medal 2010 for his role in the formulation of the modern theory of non-liner elliptic partial differential equations and for mentoring numerous students and post-docs in this area.

4.2 From FUNDP

Faculty position at the University of Namur

The University of Namur (FUNDP) has vacancy for a full-time tenured-track faculty position (m/f) at the Department of Mathematics (Applied Mathematics - Complex Systems).

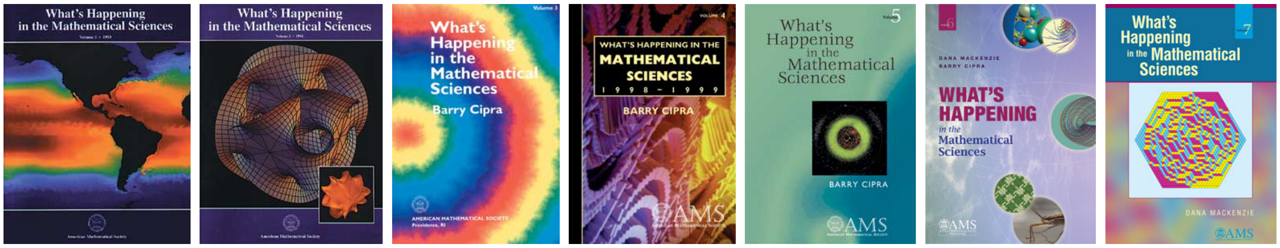
Preference should be given to a candidate with expertise in STATISTICS and/or in SYSTEMS and CONTROL.

The application DEADLINE is September 13, 2010. The position starts in January 1, 2011.

For a more complete description, see below and the web site http://www.fundp.ac.be/universite/jobs/emploi.2010-06-29.7824267395

5 History, maths and art, fiction, jokes, quotations...

What's happening in the mathematical sciences, vol. 7 *Dana Mackenzie*, American Mathematical Society, 2009 (127 p.), soft cover, ISBN 978-0-8218-4478-6, US$15.95.



Covers of volumes 1-7 of *What's happening in the mathematical sciences*

What you see above are the covers of the first seven volumes of the AMS series *What's happening in the mathematical sciences*. I have here in front of me the latest volume 7 from 2009. The first five volumes appeared respectively in 1993, 1994, 1996, 1999, and 2002. They were all authored by Barry Cipra. Volume six is from 2006 and Dana Mackenzie entered the picture because Barry Cipra could not finish that one. With the latest volume the responsibility has completely been transferred to Dana Mackenzie.

Barry Cipra

The purpose of the series is to give on a regularly basis short introductions for a broad mathematical audience to recent evolutions in mathematics. There are around 10 topics per issue, each one taking about 10 pages, which are usually richly illustrated. Excerpts from some of the volumes can be found as google books on the web.

To come to volume seven, there are 9 contributions. They are all showcases that appeal to any reader with a general mathematical interest. There is not to much mathematics, in the sense that formulas are almost completely absent. But on the other hand it takes a minimal mathematical skill to understand what is actually going on. Indeed, a formula pops up once in a while, and then of course one should be able to understand series, matrices, summations, products, equations, coordinate systems, groups, etc. Nothing fancy, yet not for a mathematical illiterate either. Although not explicit, the chapters are written like one would write an interview with the mathematical experts for the problem being discussed. It is clear the Dana Mackenzie has talked to the researchers and the text is a sedimentation of these interviews, not a summary of published papers.

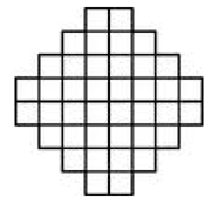
Dana Mackenzie

It is not a coincidence that new results emerge where previously unrelated areas meet. That is for example the case in the first contribution: *A new twist in knot theory*. The triggering result is that it was proved in 2006s by E. Ghys that a modular knot is topologically identical to a Lorenz knot. The first one living in number theory and the second one in dynamical systems. The article sketches the history and possible consequences of the result, even linking it with the Riemann Hypothesis (discussed in volumes 4 and 5 of this series). It is worthwhile to have a look at the beautifully illustrated online text by E. Ghys *Lorenz and modular flows: a visual introduction* at *http://www.ams.org/featurecolumn/archive/lorenz.html*, one of the AMS featured columns. These columns have a purpose very close to the purpose of the series under review.

The second article *Error-term roulette and the Sato-Tate conjecture* is again about number theory (clearly deterministic) but now interfering with probability theory (obviously stochastic). For example the Prime Number Theorem (1898) which roughly says that the probability of a number being prime is inversely proportional to the number of digits in the number. But such statements only hold 'on the average'. This links up again with the Riemann Hypothesis. The Sato-Tate conjecture is also about counting: how much deviates the number of points on an elliptic curve from the mean value, and what is the distribution of these deviations. That problem was solved in 2006. That proof is the result of an interaction of three major machineries: L-functions, automorphic forms and Galois representation theory, following a road-map pointed out by Jean-Pierre Serre. The article is mainly explaining this link between the three pillars.

The fifty-one percent solution is about the surprising observation that when tossing a coin, tossed vigorously enough and caught in midair has about a 51% chance of landing with the same face up as started with. This article describes the analysis of two Stanford researchers, Persi Diaconis and Susan Holmes, and Richard Montgomery from UC at Santa Cruz, analysing the dynamics of the normal vector on the coin when tossed. Their finding were published in 2007 in *Siam Review* Volume 49, Issue 2, pp. 211-235, that confirmed the observation.

Given a chessboard with two diagonally opposite corner squares removed, can you cover the remaining part with 31 dominos. The answer is no, and there is an easy proof. This is the prototype of the problems about random tilings considered in *Dominos, anyone*. If there is indeed a feasible solution, how many different ones do there exist, and how about a hexagon tiled with lozenges (see cover page of this volume) or rhombi; or tiling an Aztec diamond? Investigating the random tilings of Aztec diamonds of fixed size with dominos of decreasing size gives connections with statistical physics describing crystalline of liquid phases.


Aztec diamond

Not seeing is believing brings us at the verge of Harry Potter's invisibility cloak or Star Trek cloaking devises. In fact a first prototype of such a cloak was built in 2006 at Duke university. In fact invisibility for certain wavelengths in tomography, this is a serious problem. This is a contribution where physics dominates with e.g. Maxwell equations and magnetic fields, yet there is some interesting mathematics underneath.

The minimal model program by S. Mori (Fields Medal 1990) is an active research topic in algebraic geometry. It looks for the "simplest" birational model, i.e., a version of any complex variety that still has the same function space defined on it. *Getting with the (Mori) program* illustrates the historical background from the "Italian School" to the progress made in extending results from dimension 2 to higher dimensions.

In *The cook that time couldn't erase* the remarkable story is told about a palimpsest, i.e., a parchment book of which the pages had been scraped, turned over 90 degrees and written over. The original text was an account of writings by Archimedes, which had been transformed into a prayer book by monks in 1229. When it was sold during an auction in 1998, it was in a terrible state. However science as carefully recovered the original text (pet name "Archie") which is now publically available at *www.archimedespalimpsest.org* with the explanation of the recovery project and the tools used. This is an "Indiana Jones story" from real life.

The story in *Charting a 248-dimensional world* is about the Lie group E_8, with its 248 parameters, it is the largest of the exceptional groups in Killing's classification. Writing it as a combination of its irreducible representations requires a matrix that uses 60 gigabytes of data, 60 times the amount of data in the human gnome. Thanks to ingenious programming, the character map of E_8 was completed in 2007.

The last contribution is again applied. The title *Compressed sensing makes every pixel count* says indeed what the main idea is behind the buzz-word "compressed sensing". Almost everyone is making digital pictures now and it is also familiar that these big matrices of pixels (the number of mega pixels is considered to be a measure of quality of the camera) are then compressed to e.g., a jpeg format so that it tikes much less space to store. The purpose of compressed sensing is to reduce the number of pixels that are sensed, and then mathematically recover a good image from even an undersampled observation that does cover the information content of the image. For example, with only few characteristics, it is possible to reconstruct a human face. The "single-pixel camera" captures only few randomly chosen pixels to reconstruct the image. Hence mathematicians and engineers start to think beyond Shannon's sampling theorem.

Books, articles and websites like this one are multiplying fast and it is a fortunate sign that mathematicians take the trouble of explaining their work to a more general public, hopefully attracting young enthusiastic students and, why not, convince people of the importance of funding their work.

Adhemar Bultheel

The Cauchy-Kovalevsky Extension Theorem in discrete Clifford Analysis

UNIVERSITEIT GENT

FACULTY OF ENGINEERING

Hilde De Ridder*, supervisors Hennie De Schepper, Frank Sommen

Clifford Research Group

Department of Mathematical Analysis, Faculty of Engineering, Ghent University

Discrete framework

- Equidistant lattice with variable mesh width $h > 0$:

$$\mathbb{Z}_h^m = \{(\ell_1 h, \ell_2 h, \dots, \ell_m h) : (\ell_1, \ell_2, \dots, \ell_m) \in \mathbb{Z}^m\}$$

- Forward/backward basis vectors $\mathbf{e}_j^{\pm}$ $(j = 1, \dots, m)$

$$\mathbf{e}_j^- \mathbf{e}_\ell^- + \mathbf{e}_\ell^- \mathbf{e}_j^- = 0$$
$$\mathbf{e}_j^+ \mathbf{e}_\ell^+ + \mathbf{e}_\ell^+ \mathbf{e}_j^+ = 0$$
$$\mathbf{e}_j^+ \mathbf{e}_\ell^- + \mathbf{e}_\ell^- \mathbf{e}_j^+ = \delta_{j\ell}$$

- Forward/backward difference operators

$$\Delta_j^+[u] = \frac{u(\cdot + h\mathbf{e}_j) - u(\cdot)}{h}, \qquad \Delta_j^-[u] = \frac{u(\cdot) - u(\cdot - h\mathbf{e}_j)}{h}$$

- Discrete Dirac operator

$$D = \sum_{j=1}^m \left(\mathbf{e}_j^+ \Delta_j^+ + \mathbf{e}_j^- \Delta_j^-\right) = \sum_{j=1}^m \partial_j, \qquad D^2 = \Delta^*$$

- Discrete monogenic function f: $Df = 0$
- Discrete vectorvariable

$$X = \sum_{j=1}^m \left(\mathbf{e}_j^+ X_j^- + \mathbf{e}_j^- X_j^+\right) = \sum_{j=1}^m \xi_j$$

- Skew Weyl relations:

$$\begin{cases} \Delta_j^+ X_j^+ - X_j^- \Delta_j^- = 1 \\ \Delta_j^- X_j^- - X_j^+ \Delta_j^+ = 1 \end{cases} \Rightarrow \begin{cases} \partial_j\, \xi_j - \xi_j\, \partial_j = 1 \\ \partial_k\, \xi_j + \xi_j\, \partial_k = 0 \ (k \neq j) \end{cases}$$

- Discrete Euler operator: $DX + XD = 2E + m$
- Discrete homogeneous polynomials P_k: $EP_k = kP_k$

$$\xi_j[1](x_j) = x_j\left(\mathbf{e}_j^+ + \mathbf{e}_j^-\right)$$
$$\xi_j^{2n}[1](x_j) = \left(x_j^2 + nhx_j(\mathbf{e}_j^+\mathbf{e}_j^- - \mathbf{e}_j^-\mathbf{e}_j^+)\right) \prod_{i=1}^{n-1} (x_j^2 - h^2 i^2)$$
$$\xi_j^{2n+1}[1](x_j) = x_j\left(\mathbf{e}_j^+ + \mathbf{e}_j^-\right) \prod_{i=1}^{n} (x_j^2 - h^2 i^2)$$

- Involution $\hat{\ }$: $\hat{\xi}_1 = -\xi_1$, $\hat{\xi}_j = \xi_j$ $(j = 2, \dots, m)$, $\widehat{ab} = \hat{a}\hat{b}$

Example 1: CK $[\delta_0(x_2)]$ in two dimensions $(m = 2)$

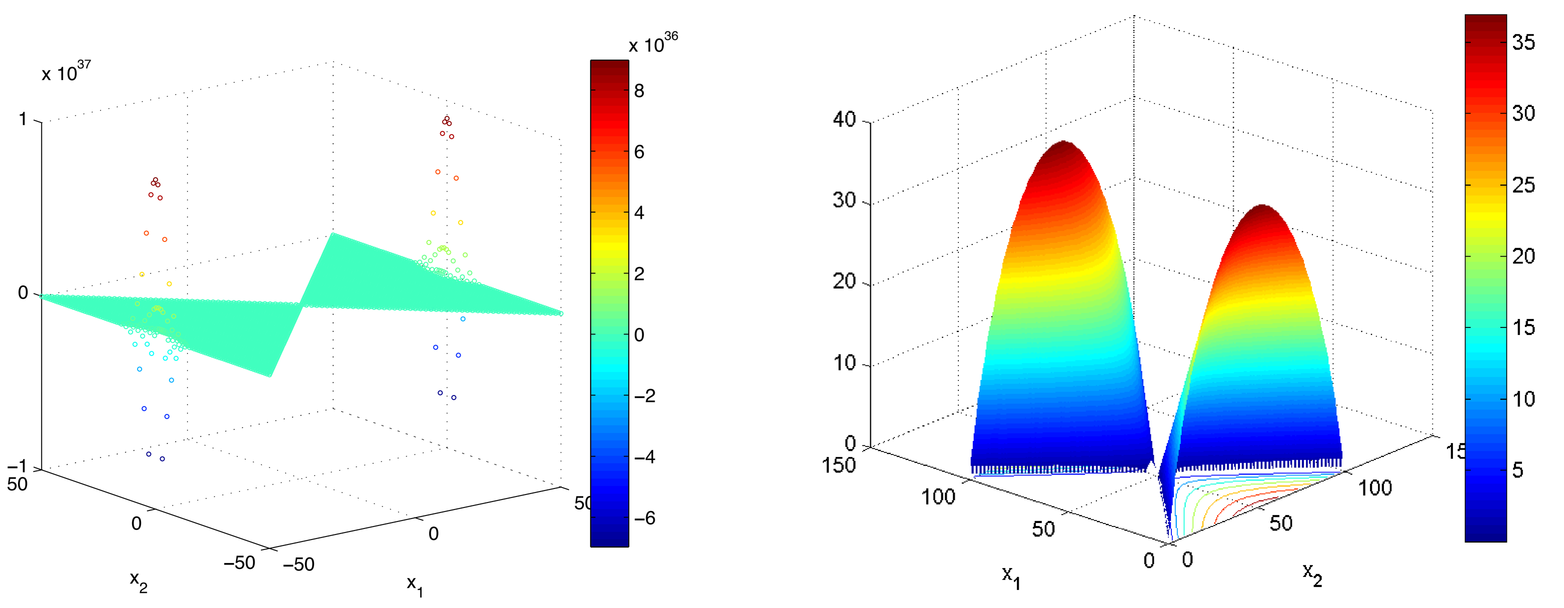


Figure: Scalar part of CK $[\delta_0(x_2)]$ and $\log_{10}$ plot of the absolute values; $h = 1$

- The discrete delta function

$$\delta_0(x_2, \dots, x_m) = \begin{cases} 1, & (x_2, \dots, x_m) = (0, \dots, 0) \in \mathbb{Z}_h^{m-1} \\ 0, & (x_2, \dots, x_m) \neq (0, \dots, 0) \in \mathbb{Z}_h^{m-1} \end{cases}$$

  ▷ basic building block
  ▷ continuous delta function: not real-analytic $\Rightarrow$ no CK extension

- For $(x_1, x_2) \in \mathbb{Z}_h^2$, the scalar part of CK $[\delta_0(x_2)](x_1, x_2)$ is

$$\begin{cases} 1 + \displaystyle\sum_{k=1}^{|x_1/h|} \frac{1}{h^{2k}(k!)^2}\, x_1^2 \prod_{m=1}^{k-1} (x_1^2 - m^2h^2), & x_2 = 0 \\ (-1)^{|x_2/h|} \displaystyle\sum_{k=|x_2/h|}^{|x_1/h|} \frac{1}{h^{2k+1}(k - \frac{x_2}{h})!\,(k + \frac{x_2}{h})!}\, x_1^2 \prod_{m=1}^{k-1} (x_1^2 - m^2h^2), & x_2 \neq 0 \end{cases}$$

- For $h \to 0$ the values of CK$[\delta_0]$ diverge

Discrete CK extension

Discrete CK extension Theorem

- $f(x_2, \dots, x_m)$: discrete function, defined on $\mathbb{Z}_h^{m-1}$ and taking values in the algebra over $\{\mathbf{e}_2^+, \mathbf{e}_2^-, \dots, \mathbf{e}_m^+, \mathbf{e}_m^-\}$

- Cauchy-Kovalevsky extension of f:
$\exists!$ discrete **monogenic** function $F(x_1, x_2, \dots, x_m)$, defined on $\mathbb{Z}_h^m$ and taking values in the algebra over $\{\mathbf{e}_1^+, \mathbf{e}_1^-, \dots, \mathbf{e}_m^+, \mathbf{e}_m^-\}$, s.t. $F|_{x_1=0} = f$.
Denote $D' = \sum_{j=2}^m \partial_j$, then

$$\text{CK}[f](x_1, x_2, \dots, x_m) = \sum_{k=0}^{\infty} \frac{\xi_1^k[1](x_1)}{k!} f_k(x_2, \dots, x_m) \qquad (*)$$

with $f_0 = f$, $f_{k+1} = (-1)^{k+1} D' f_k$

- No conditions on the function f:

$$\xi_1^{2k+1}[1](x_1) = 0 \quad \text{for } k \geqslant \frac{|x_1|}{h}$$
$$\xi_1^{2k}[1](x_1) = 0 \quad \text{for } k \geqslant \frac{|x_1|}{h} + 1$$

$\Rightarrow \forall (x_1, \dots, x_m) \in \mathbb{Z}_h^m$, the series $(*)$ reduces to a finite sum

Fueter polynomials

- Discrete Fueter polynomials of degree k constitute a basis for $\mathcal{M}_k^{(m)}$:

$$\text{CK}\,[\xi_2^{\alpha_2} \dots \xi_m^{\alpha_m}] \text{ with } |\underline{\alpha}| = k$$

- $z_i = \text{CK}[\xi_i] = \xi_i - \xi_1$ $(i = 2, \dots, m)$ and

$$V_{\ell_1, \dots, \ell_k} = \frac{1}{k!} \sum_{\pi(\ell_1, \dots, \ell_k)} \text{sgn}(\pi)\, z_{\pi(\ell_1)} \dots z_{\pi(\ell_k)} \quad (2 \leqslant \ell_j \leqslant m)$$

  ▷ sum over all distinguishable permutations π of $(\ell_1, \dots, \ell_k)$
  ▷ every second z_j in each term is replaced by $\hat{z}_j$, $j = 2, \dots, m$

- For $\underline{\alpha} = (\alpha_2, \dots, \alpha_m) \in \mathbb{N}^{m-1}$: $\underline{\alpha} = (\ell_1, \ell_2, \dots, \ell_k) \in \{2, \dots, m\}^k$ s.t. $k = |\underline{\alpha}|$, $\ell_i \leqslant \ell_j$ for $i < j$ and $\#j$ in $(\ell_1, \ell_2, \dots, \ell_k)$ is α_j. Then

$$\text{CK}\,[\xi_2^{\alpha_2} \dots \xi_m^{\alpha_m}] = \alpha_2! \dots \alpha_m!\, V_{\ell_1, \dots, \ell_k}$$

Example 2: CK $[\exp(x_2)]$ in two dimensions $(m = 2)$

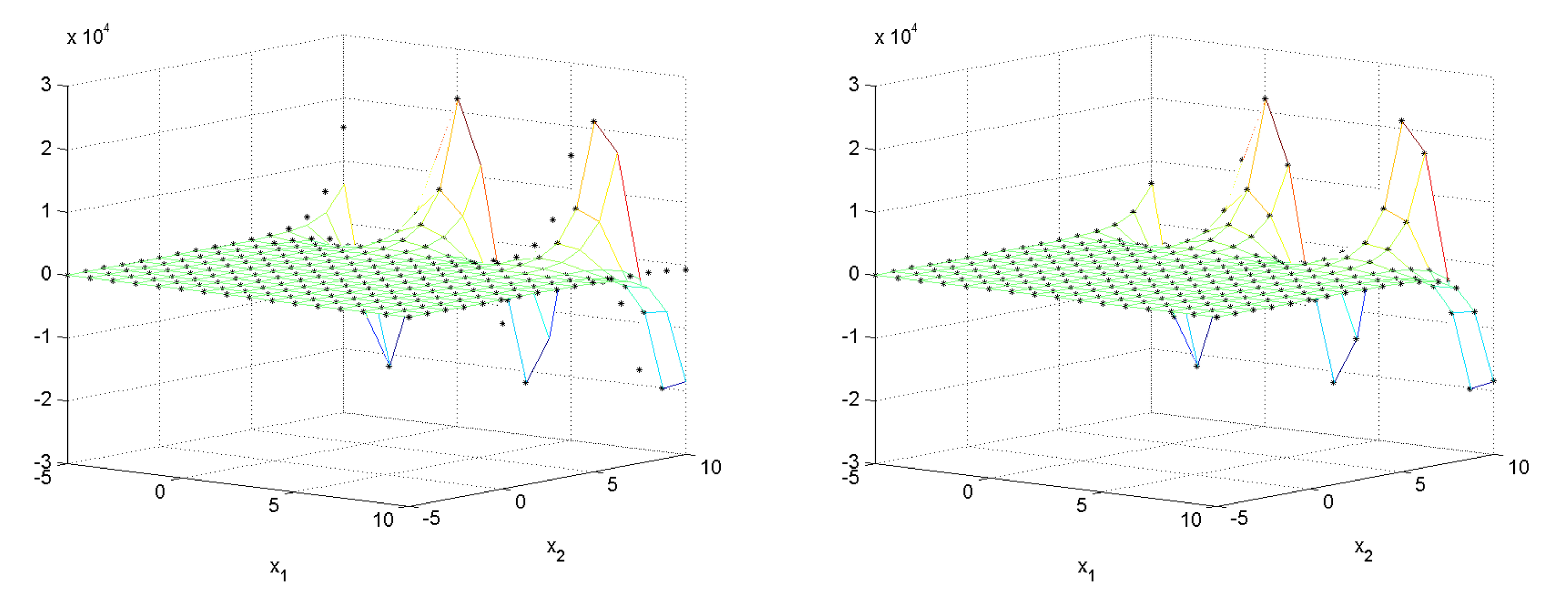


Figure: Continuous and discrete scalar part CK $[\exp(x_2)](x_1, x_2)$ for $h = 1$, $h = \frac{1}{10}$

- The discrete exponential function $\exp(x_2)$
  ▷ restriction of $\exp(x_2)$ to $\mathbb{Z}_h$
  ▷ continuous $\exp(x_2)$ is real-analytic:

$$\text{CK}\,[\exp(x_2)](x_1, x_2) = \exp\left(-x_1 \mathbf{e}_1 \partial_{\underline{x}}'\right) [\exp(x_2)]$$

- For $(x_1, x_2) \in \mathbb{Z}_h^2$, the scalar part of CK $[\exp(x_2)](x_1, x_2)$ is

$$\exp(x_2) \left[1 + \sum_{k=1}^{\left|\frac{x_1}{h}\right|} (-1)^k \frac{\lambda_{k,h}}{(2k)!\, h^{2k}}\, x_1^2 \prod_{m=1}^{k-1} (x_1^2 - m^2h^2) \right]$$

with $\lambda_{k,h} = \displaystyle\sum_{m=-k}^{k} (-1)^m \binom{2k}{k-m} \exp(mh)$

- For $h \to 0$, the discrete CK extension tends to the continuous CK extension

*The author acknowledges support by the institutional grant no. B/10675/02 of Ghent University (BOF).

Hilbert space compression of groups

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Background (1)

The class of finitely generated groups that are uniformly embeddable into a Hilbert space was introduced by Gromov [2]. This class is very interesting: it has very nice permanence properties, e.g. direct sums, direct limits, amalgamated free products... and it is very large: in fact, one has to be very creative to find a countable group which is not uniformly embeddable! It is quite shocking that a property which is so easily satisfied has so many strong consequences. In [5] and [4] it is proven that uniformly embeddable groups satisfy both the Novikov and the coarse Baum-Connes Conjecture.

In the domain of isometric actions of groups on Hilbert spaces, one comes across groups that satisfy the property of Haagerup. This large class of groups contains all amenable groups. The class and its permanence properties have been very well studied [1].

Background (2)

Quantifying *how uniformly embeddable* a group really is, proves to be very fascinating and opens a new domain of research: that of Hilbert space compression.

Analogously, quantifying *how Haagerup* a group really is, leads to the notion of equivariant Hilbert space compression.

In all that follows, we only deal with finitely generated groups that are equipped with the word length metric relative to a finite symmetric generating subset S. All of the following definitions will be independent of the choice of the generating subset.

Uniform Embeddability (Classical definition)

Definition 1.1. *A finitely generated group G is* uniformly embeddable in a Hilbert space*, if there exist a Hilbert space $\mathcal{H}$, a map $f : G \to \mathcal{H}$ and non-decreasing functions $\rho_-, \rho_+ : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{t\to\infty} \rho_-(t) = +\infty$ such that $\forall x, y \in G$:*

$$\rho_-(d(x,y)) \leq d(f(x), f(y)) \leq \rho_+(d(x,y)).$$

The map f is called a uniform embedding of G in $\mathcal{H}$.

The above definition can easily be generalized to the case of an arbitrary metric space.

Hilbert space compression

Definition 1.2. *[3] Let $f : G \to \mathcal{H}$ be a uniform embedding of a finitely generated group. The compression $R(f)$ of f is the supremum of $\alpha \in [0,1]$ such that there exist $C > 0, D \geq 0$ satisfying that $\forall x, y \in G$:*

$$(1/C)d(x,y)^\alpha - D \leq d(f(x), f(y)) \leq Cd(x,y) + D.$$

Definition 1.3. *The Hilbert space compression of G is the* $\sup\{R(f) \mid f$ *is a uniform embedding of G into a Hilbert space* $\}$.

Quantification idea 1: *The higher the compression, the more* uniformly embeddable *G is.*

Uniform Embeddability (Practical definition)

We investigate the behaviour of the Hilbert space compression when taking free products, direct limits, extensions, . . . The proofs heavily rely on the following equivalent definition for uniform embeddability.

Definition 1.4. *A finitely generated group G is* uniformly embeddable in a Hilbert space*, if and only if for every $n \in \mathbb{N}_0$ there exist $S_n > 0$ and a Hilbert space valued map $\xi_n : G \to \mathcal{H}, x \to \xi_n^x$ such that $\| \xi_n^x \| = 1$ for all $x \in G$ and such that*

1. $\| \xi_n^x - \xi_n^{x'} \| \leq \frac{1}{n}$ provided $d(x, x') \leq \sqrt{n}$,
2. $\| \xi_n^x - \xi_n^{x'} \| \geq 1$ provided $d(x, x') \geq S_n$.

Quantification idea 2: *The slower $n \mapsto S_n$ increases, the more* uniformly embeddable *G is.*

Free products and HNN-extensions

Theorem 1.5. *Denote the Hilbert space compressions of finitely generated groups G_1 and G_2 by α_1 and α_2 respectively. Let F be a common finite subgroup. The Hilbert space compression α of the amalgamated free product $G = G_1 *_F G_2$ satisfies*

$$\min(\alpha_1, \alpha_2, 1/2) \leq \alpha \leq \min(\alpha_1, \alpha_2).$$

Theorem 1.6. *Consider $G := HNN(H, F, \theta)$ where both F and $\theta(F)$ are finite index subgroups of the finitely generated group H. Equip H with the induced metric d_{in} from G. Denoting the Hilbert space compressions of (H, d_{in}) and G by α_1 and α respectively, we get*

$$\alpha_1/3 \leq \alpha \leq \alpha_1.$$

Group extensions

Theorem 1.7. *Assume that Γ is a finitely generated group that fits in a short exact sequence*

$$1 \to H \to \Gamma \xrightarrow{\pi} G \to 1.$$

If G has polynomial growth and if H with the induced metric from Γ has compression α, then the compression of Γ is at least $\alpha/4$.

Theorem 1.8. *Assume that Γ is a finitely generated group that fits in a short exact sequence*

$$1 \to H \to \Gamma \xrightarrow{\pi} G \to 1.$$

If G is a hyperbolic group in the sense of Gromov and if H, with the induced metric from Γ, has Hilbert space compression α, then the Hilbert space compression of Γ is at least $\alpha/5$.

Equivariant Hilbert space compression

Definition 1.9. *The equivariant Hilbert space compression of G is the $\sup\{R(f) \mid f$ is a uniform embedding of G into some Hilbert space $\mathcal{H}$ such that f is G-equivariant relative to the left multiplication action of G on itself and some affine isometric action of G on $\mathcal{H}\}$.*

We investigate the behaviour of the equivariant compression when taking free products and HNN-extensions. The idea of the proof is based on the concept of conditionally negative definite functions on groups.

Free products (Equivariant case)

Theorem 1.10. *Let G_1 and G_2 be finitely generated groups with equivariant Hilbert space compressions equal to α_1 and α_2 respectively. Denote $G = G_1 *_F G_2$ an amalgamated free product where F is a finite subgroup of both G_1 and G_2. If α denotes the equivariant Hilbert space compression of G, then*

1. $\alpha = 1$ if F is of index 2 in both G_1 and G_2,
2. $\alpha = \alpha_1$ if $F = G_2$ and $\alpha = \alpha_2$ if $F = G_1$,
3. $\alpha = \min(\alpha_1, \alpha_2, 1/2)$ otherwise.

HNN-extensions (Equivariant case)

Theorem 1.11. *Let H be a finitely generated group with equivariant Hilbert space compression α_1. Assume that F is a subgroup of H and that $\theta : F \to H$ is a group monomorphism such that the group generated by $\theta(F) \cup F$ is finite. Denoting the equivariant Hilbert space compression of $HNN(H, F, \theta)$ by α, we get*

1. $\alpha = 1$ whenever $F = H$,
2. $\alpha = \min(\alpha_1, 1/2)$ otherwise.

References

[1] P. A. Cherix, M. Cowling, P. Jolissaint, P. Julg, A. Valette, Groups with the Haagerup Property, Progress in Mathematics 197, 2001.

[2] M. Gromov, Asymptotic invariants of infinite groups, Geometric Group Theory (A.Niblo and M. Roller, eds.), London Mathematical Society Lecture Notes 182 (1993), 1–295.

[3] E.Guentner, J.Kaminker, 'Exactness and uniform embeddability of discrete groups', *Journal of the London Mathematical Society* 70, no.3 (2004), 703–718

[4] G. Skandalis, J. L. Tu, and G. Yu, Coarse Baum-Connes conjecture and groupoids, Topology 41 (2002), 807–834.

[5] G. Yu, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, Invent. Math. 139, no. 1 (2001), 201–240.

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One-dimensional wrinkling of thin membranes

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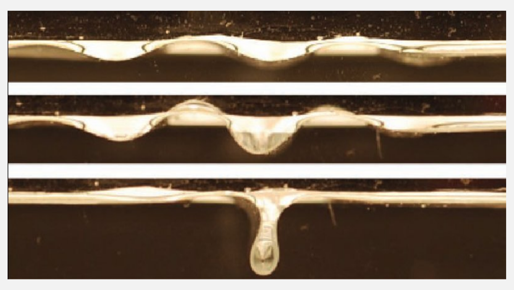
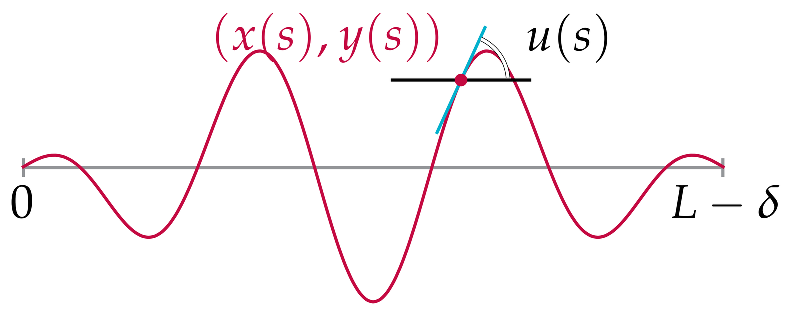
Physical experiments [3]



A film, initially flat, is laid on a substrate such as water and compressed horizontally at both edges. It first starts to wrinkle, taking a sinusoidal form, but, for larger compressions, seems to concentrate at the centre.

Steps leading to the limit problem $\delta \to 0$:

1. Boundedness of $\{\mathcal{E}(u_\delta)\}_\delta$ for any $(u_\delta)_{\delta>0}$ minimizers.
2. Inequality $\|u_\delta\|_X^2 \le 2\mathcal{E}(u_\delta) + KL^3 \max\{1, \|u_\delta\|_\infty^2\}$, implies $u_\delta \rightharpoonup u_0$ but, as $\int_0^L 1 - \cos u_\delta = \delta$, $u_0 = 0$.
3. The sequence $(u_\delta/\sqrt{\delta})$ is also bounded in H_0^1; so $u_\delta/\sqrt{\delta} \rightharpoonup u^*$ with $\|u^*\|_{L^2} = \sqrt{2}$.
4. Let us call α_δ and β_δ the two Lagrange multipliers appearing in the Euler-Lagrange equation (see (3)). By a good choice of "test functions" v, we can deduce estimates on these constants, permitting us to divide this equation by $\sqrt{\delta}$, giving the limit case (4).



We assume that there is no variation other than those in the compression direction ⇒ one-dimensional parametrization.
Minimization of the energy due to :

- folds (which is measured by its curvature);
- potential energy due to the displacement of the substrate underneath.

⇒ We are seeking for minima of the functional

$$\mathcal{E} : X \to \mathbb{R} : u \mapsto \tfrac{1}{2}\int_0^L |u'|^2\,\mathrm{d}s + \tfrac{1}{2}K\int_0^L (y_u(s))^2 \cos u(s)\,\mathrm{d}s \tag{1}$$

where $y_u(s) = \int_0^s \sin u(t)\,\mathrm{d}t$, K is a constant relative to the substrate and X describes the space of admissible functions:

$$X = \Big\{ u \in H_0^1(]0, L[; \mathbb{R}) \;\Big|\; \int_0^L \cos u(s)\,\mathrm{d}s = L - \delta \quad\text{and}\quad \int_0^L \sin u(s)\,\mathrm{d}s = 0 \Big\}. \tag{2}$$

Going to the limit $\delta \to 0$...

Euler-Lagrange equation of the problem:

$$\partial\mathcal{E}(u_\delta)\cdot v + \alpha_\delta \int_0^L \sin u_\delta \cdot v + \beta_\delta \int_0^L \cos u_\delta \cdot v = 0 \tag{3}$$

As $\delta \to 0$, we obtain:

$$\int_0^L u^{*\prime} v' + K \int_0^L \int_0^{\cdot} u^* \int_0^{\cdot} v + \alpha \int_0^L u^* v + \beta \int_0^L v = 0 \tag{4}$$

where $\alpha = \lim_{\delta\to 0} \alpha_\delta$ and $\beta = \lim_{\delta\to 0} \beta_\delta/\sqrt{\delta}$ (up to subsequences).

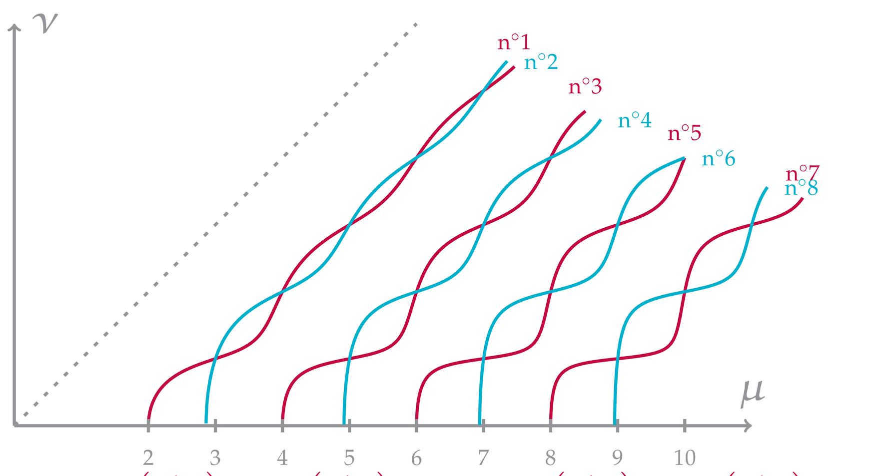
This gives then the eigenvalue problem

$$\begin{cases} w^{iv} - \alpha w'' + Kw = 0 \\ w(0) = w(L) = 0 \\ w'(0) = w'(L) = 0 \end{cases} \tag{5}$$

(where $w' = u^*$) with the additional constraint

$$\int_0^L w'^2 = 2. \tag{6}$$

The equation (5) admits nontrivial solutions only if $\alpha < -2\sqrt{K}$; the characteristic polynomial admits then two pairs of purely imaginary complex roots. For convenience reasons, let us denote their moduli by $\frac{\pi}{L}\mu$ and $\frac{\pi}{L}\nu$ with, w.l.o.g. $\mu > \nu$.



$\mu \sin\left(\frac{\pi}{2}\mu\right)\cos\left(\frac{\pi}{2}\nu\right) - \nu \sin\left(\frac{\pi}{2}\nu\right)\cos\left(\frac{\pi}{2}\mu\right) = 0$
$\mu \cos\left(\frac{\pi}{2}\mu\right)\sin\left(\frac{\pi}{2}\nu\right) - \nu \cos\left(\frac{\pi}{2}\nu\right)\sin\left(\frac{\pi}{2}\mu\right) = 0$

By successively imposing the boundary conditions on a general solution w, we get the two equations in μ and ν for which a part of the solutions is drawn.

1D or 2D space of solutions for equation (5)?

- 2D space ⇔ (μ, ν) in intersection ⇔ $(\mu, \nu) \in \mathbb{N} \times \mathbb{N}$ with $\mu + \nu$ even.
- 1D space ⇔ (μ, ν) in (only) one curve.

What about eigenvalues? We get $-\alpha = \frac{\pi^2}{L^2}(\mu^2 + \nu^2)$. Thus, for fixed K, the lowest eigenvalue $-\alpha$ is found for (μ, ν) lying on the curve closest to the diagonal (depending on K, it is curve n°1 or n°2).

Possible degeneracy of eigenvalues: e.g. for the first one, for all $k \in \mathbb{N}$, $K = \frac{\pi^4}{L^4}k^2(k+2)^2$ will give a 2D first eigenspace.

Some properties on the "limit solutions"

- **Form of the solutions:** We have $w(s) = \hat{w}\left(\frac{s}{L} - \frac{1}{2}\right)$ with, in case 1D space,

$$\hat{w}(t) = A\Big(\cos\left(\tfrac{\pi}{2}\mu\right)\cos(\pi\nu t) - \cos\left(\tfrac{\pi}{2}\nu\right)\cos(\pi\mu t)\Big)$$

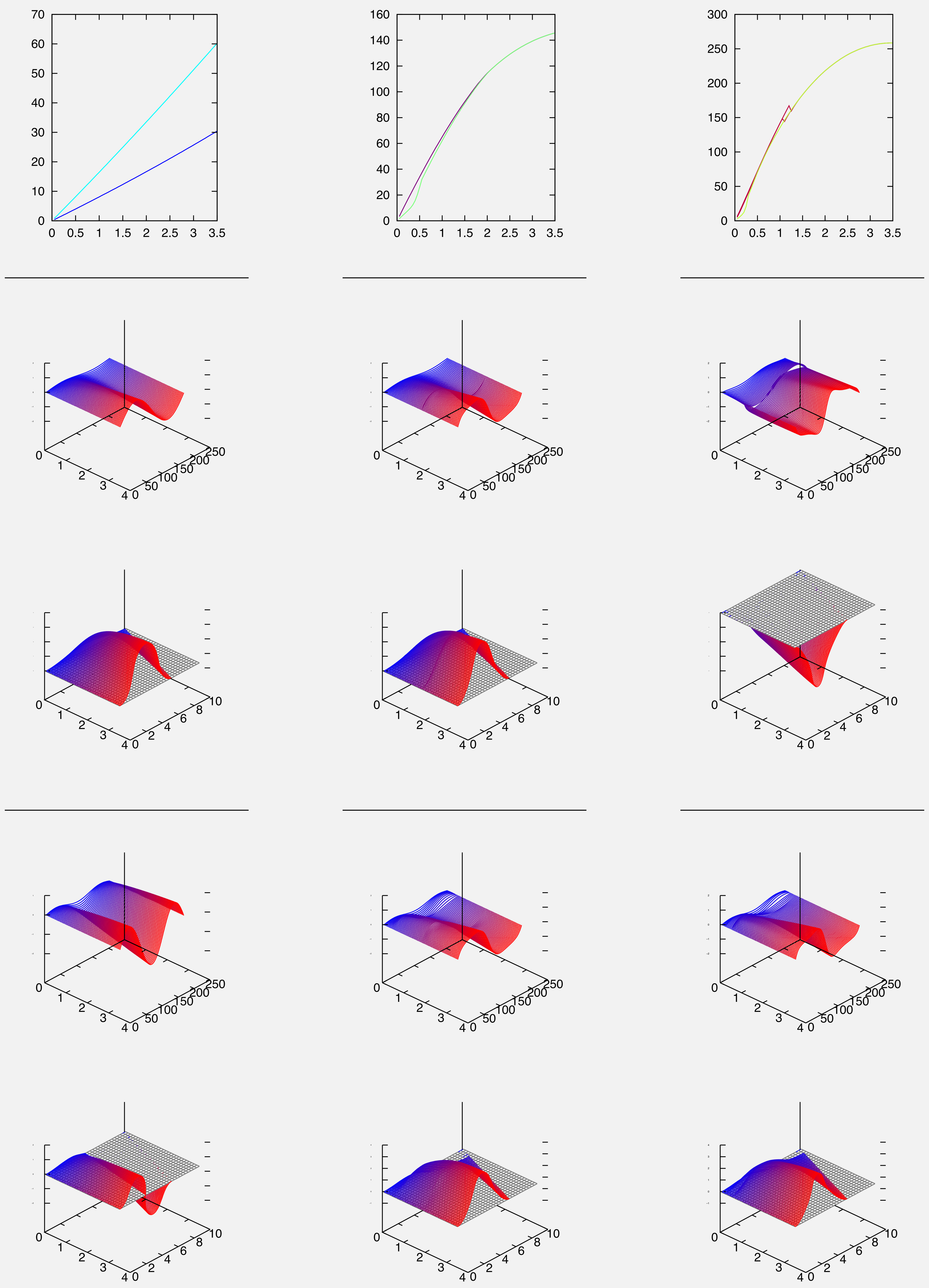
$$\hat{w}(t) = A\Big(\sin\left(\tfrac{\pi}{2}\mu\right)\sin(\pi\nu t) - \sin\left(\tfrac{\pi}{2}\nu\right)\sin(\pi\mu t)\Big)$$

  where A is determined (up to sign) by (6).
  In case of 2D space, $\hat{w}$ can be any linear combination of two functions like above, with adjusted coefficients; fixing constraint (6) selects an ellipse in the space of solutions.
- **Symmetry:** if $w(s)$ is a solution of (5) then $w(L-s)$ is also solution; it is $\pm w(s)$ in case 1D-space and can be described from $w(s)$ in case 2D-space.
- **Parity** of all solutions in case 1D-space: oddness on red curves, evenness on blue ones.
- **Number of roots** depending on μ and ν: in case 1D-space,
  - At least $n-1$ inner roots when being on the n^{th} curve.
  - Increases by 2 after a crossing for the curve going below. It seems that each newly created root comes from an edge.
  - All roots are simple.

  This behaviour is found (at least numerically) also for solutions living in the 2D spaces.

Numerical experiments: continuation algorithm applied to the "limit solutions"

Plot of the energies for the first two solutions for $K = k^2\frac{\pi^4}{L^4}$ with, respectively $k = 0.01$, 2, 3. Below are the curves u obtained by continuation, followed by the resulting curves (x, y). (Length L is fixed to 10 and δ varies from 0.05 to 3.5 by step 0.05.)



Future work

- Evolution of the "degeneracy points" (2D space) when δ grows.
- Study of the equation as $K \to 0$ (related to the Euler-Bernoulli elastica problem). Continuation algorithm applied to elastica curves.
- Study of the equation for large values of K.
- Variable coefficients in equation (5).

References

[1] H.-G. Grunau. Positivity, change of sign and buckling eigenvalues in a one-dimensional fourth order model problem. *Advances in differential equations*, 7(2):177–196, 2002.
[2] M. A. Peletier. Sequential buckling: a variational analysis. *SIAM J. Math. Anal.*, 32(5):1142–1168, September 2000.
[3] L. Pocivavsek, R. Dellsy, A. Kern, S. Johnson, B. Lin, K. Y. C. Lee, and E. Cerda. Stress and fold localization in thin elastic membranes. *Science*, 320:912–916, May 2008.
[4] J. A. Zasadzinski, J. Ding, H. E. Warriner, A. J. Bringezu, and A. J. Waring. The physics and physiology of lung surfactants. *Current Opinion in Colloid and Interface Science*, 6(5):506–513, November 2001.

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Properties and applications of quantifiable domains

Topology meets computer science

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Domain Theory

- Partially ordered set [illegible]
- [illegible] Closed under directed suprema
- [illegible]
- [illegible] Axioms of approximation
- [illegible]
- [illegible] Countable base
- [illegible] Base of compact elements
- [illegible]
- [illegible] original domain order
- [illegible]

Quantifiability

- [illegible] Reflexivity and triangle inequality
- [illegible]
- [illegible]

Semantics

- [illegible] Meaning of computer program
- [illegible] of type is domain
- [illegible] is undefined object
- [illegible] of a program
  function from input to output domain
- [illegible] functions are [illegible]
- [illegible]
- [illegible] (LFP)
- [illegible] of recursive function via LFP of functional
- [illegible] Factorial

Approachability

- [illegible]
  Properties
- [illegible]
  Properties

MOQA semantics

- MOQA designed for [illegible]
- No [illegible]
- Data types are [illegible]
- [illegible]
- [illegible] functions [illegible]
- RB-preserving functions are [illegible]
  ACET(P ∘ Q) = ACET(P) + ACET(Q)
- Prediction of [illegible] and [illegible]
- Work in progress
  - Develop [illegible] for MOQA
  - Design [illegible] for MOQA
  - Better understanding of [illegible]
  - Derive [illegible] from [illegible]

Approach Spaces

- [illegible] (AP) supercategory of TOP and pqMet
- Determined by [illegible]
  - [illegible] Collection of q-metrics
  - [illegible] Distance between point and subset
  - [illegible] Degree of convergence
  - [illegible] Local distances
- [illegible] of topological concepts
  - distance ↔ closure operator
  - limit operator ↔ topological convergence
  - Approach system ↔ neighbourhood system
- [illegible]
- [illegible]
- [illegible]

Complexity Space

- [illegible] with pointwise order
- [illegible]
- Ordered [illegible] and [illegible]
- [illegible]
  with weight function
- [illegible]
- Functional of [illegible] algorithm is contraction
- [illegible]
- [illegible] theorem applicable
- TODO
  - Computing fixed point for [illegible]
  - Consider only [illegible] functions

Complexity App. Space

Multiple [illegible] on the complexity space

Properties
- Close relation to [illegible]
- Better [illegible]

Properties
- Has [illegible]
- [illegible]

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Complete LR-structures on solvable Lie algebras

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1 Main Idea

1. Main Idea – We study LR-structures on Lie algebras.

An **LR-structure on a Lie algebra** $\mathfrak{g}$ over a field k is a bilinear product on $\mathfrak{g}$, satisfying the following commutativity relations

- $x \cdot (y \cdot z) = y \cdot (x \cdot z)$,
- $(x \cdot y) \cdot z = (x \cdot z) \cdot y$

for all $x, y, z \in \mathfrak{g}$, and which is compatible with the Lie product, meaning that

$$[x, y] = x \cdot y - y \cdot x$$

for all $x, y \in \mathfrak{g}$.

In particular one is interested in the question which Lie algebras admit a complete LR-structure.

An LR-structure is called complete if all right multiplications are nilpotent.

2 Background

2. Background – LR-structures arise in the study of simply transitive affine actions on Lie groups. We know that a connected and simply connected (c.s.c.) Lie group G admits a simply transitive and affine action iff its Lie algebra $\mathfrak{g}$ admits a complete left symmetric structure ([4,5,6]).

By a result of Auslander such a Lie group G (and hence $\mathfrak{g}$) has to be solvable. The converse appeared to be false ([1]) and hence one has broadened the geometric context and studied affine actions on c.s.c. nilpotent Lie groups.

In contrast to the fact that there are c.s.c. solvable Lie groups G not allowing a simply transitive affine action, all of them do admit such an action on a nilpotent Lie group.

In order to get a better understanding of such actions, one started to investigate the situation for $G = \mathbb{R}^n$ ([2,3]). Also here a translation to the Lie algebra level is possible and leads to LR-structures defined above.

An important property of Lie algebras admitting an LR-structure is that they have to be 2-step solvable. The converse is not true.

3 Nilpotent Lie Algebras

3. Nilpotent Lie Algebras – In what follows we will always assume that all Lie algebras are finite-dimensional and defined over a field k of characteristic zero.

In the study of left symmetric structures it was asked whether a complete left symmetric structure would exist automatically, once there was any such structure at all. Mizuhara proved this for the nilpotent case:

Theorem [7] *Let $\mathfrak{g}$ be a complex nilpotent Lie algebra. If $\mathfrak{g}$ admits a left symmetric structure, then $\mathfrak{g}$ also admits a complete left symmetric structure.*

We proved the analogue for LR-algebras.

Theorem *Let $\mathfrak{g}$ be a nilpotent Lie algebra over k. If $\mathfrak{g}$ admits an LR-structure then it also admits a complete LR-structure.*

4 Solvable Lie Algebras

4. Solvable Lie Algebras – Let $\mathfrak{g}$ be a Lie algebra. We denote the ideals of the lower central series by $\mathfrak{g}^1 = \mathfrak{g}$ and $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$ for $i \geq 1$.

This series stabilizes after finitely many steps, with $\mathfrak{g}^\infty = \bigcap_{i=1}^\infty \mathfrak{g}^i$. Let $\mathfrak{n} := \mathfrak{g}/\mathfrak{g}^\infty$. This is a nilpotent Lie algebra.

If $\mathfrak{g}$ is 2-step solvable, then $\mathfrak{g} = \mathfrak{g}^\infty \rtimes \mathfrak{n}$ with $[(a, x), (b, y)] = ([a, b] + \varphi(x)b - \varphi(y)a, [x, y])$, where $\varphi : \mathfrak{n} \to \mathrm{End}(\mathfrak{g}^\infty)$ with $\varphi(x)(a) = [\tilde{x}, a]$ where $\tilde{x}$ is a pre-image of x in $\mathfrak{g}$.

We have proved an analogue of Mizuhara's result for all (solvable) Lie algebras.

Main Theorem [Burde–Dekimpe–V, 2009] *Let $\mathfrak{g}$ be a Lie algebra over a field k of characteristic 0. If $\mathfrak{g}$ admits an LR-structure, then $\mathfrak{g}$ also admits a complete LR-structure.*

5 Corollaries

5. Corollaries – We proved that if $\mathfrak{g}$ admits an LR-structure, then also $\mathfrak{n}$ admits an LR-structure.

We also proved a partial converse of this.

Theorem *Let $\mathfrak{g}$ be a two-step solvable Lie algebra and assume that $\mathfrak{n} = \mathfrak{g}/\mathfrak{g}^\infty$ admits an LR-structure $(\mathfrak{n}, \cdot)$ satisfying $\mathfrak{n} \cdot \mathfrak{n} \subseteq [\mathfrak{n}, \mathfrak{n}]$. Then, this LR-structure can be lifted to an LR-structure on $\mathfrak{g} = \mathfrak{g}^\infty \rtimes \mathfrak{n}$ given by the formula $(a, x) \star (b, y) = (\varphi(x)b, x \cdot y)$.*

Corollary *Let $\mathfrak{g}$ be a two-step solvable Lie algebra with $\mathfrak{g}^\infty = \mathfrak{g}^3$. Then $\mathfrak{g}$ admits an LR-structure.*

There are two-step solvable Lie algebras with $\mathfrak{g}^\infty = \mathfrak{g}^4$ without any LR-structure, see [3]. In this sense the corollary cannot be improved.

6 Two-Step Solvable Two-Generated

6. Two-Step Solvable Two-Generated Lie Algebras – Among the two-step solvable Lie algebras with two generators are the filiform nilpotent Lie algebras of solvability class 2. For this class of Lie algebras we have constructed an explicit LR-structure in [3]. We generalized this construction to all two-generated, two-step solvable Lie algebras.

Theorem *Let $\mathfrak{g}$ be a two-generated, two-step solvable Lie algebra. Then $\mathfrak{g}$ admits a complete LR-structure.*

Suppose $\mathfrak{g}$ is generated by x and y. Then $\mathfrak{g}$ has a vector space basis consisting of x, y and a finite number of elements of the form $\mathrm{ad}(y)^k \mathrm{ad}(x)^\ell y, k \geq 0, l > 0$. The k-bilinear product defined by $L(x) = 0$ and $L(\mathrm{ad}(y)^k \mathrm{ad}(x)^\ell y) = \mathrm{ad}(y)^k \mathrm{ad}(x)^\ell \mathrm{ad}(y)$ defines an LR-structure on $\mathfrak{g}$. By our main theorem $\mathfrak{g}$ admits a complete LR-structure.

References

[1] Y. Benoist: *Une nilvariété non affine.* J. Differential Geom. **41** (1995), no. 1, 21–52.

[2] D. Burde, K. Dekimpe, S. Deschamps: *Affine action on Nilpotent Lie groups.* Forum Math. 21 (2009), no. 5, 921–934.

[3] D. Burde, K. Dekimpe, S. Deschamps: *LR-algebras.* Contemp. Math. **491** (2009), 125–140.

[4] D. Fried, W. Goldman: *Three-dimensional affine crystallographic groups.* Adv. in Math. **47** (1983), no. 1, 1–49.

[5] D. Fried, W. Goldman, M. Hirsch: *Affine manifolds with nilpotent holonomy.* Comment. Math. Helv. **56** (1981), no. 4, 487–523

[6] H. Kim: *Complete left-invariant affine structures on nilpotent Lie groups.* J. Differential Geom. **24** (1986), no. 3, 373–394.

[7] A. Mizuhara: *On a complete left symmetric algebra over a nilpotent Lie algebra.* Tensor (N.S.) **40** (1983), no. 2, 144–148.