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Recent progress on off-diagonal Ramsey numbers

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The study of Ramsey numbers was popularized by Paul Erdős in the 1930s and has grown into a cornerstone of modern-day research in combinatorics. To be precise, the Ramsey number $r(s,t)$, where $s, t > 2$, is defined as the smallest integer N such that every N-vertex graph either contains a clique of order s or an independent set of order t. Named after Frank P. Ramsey who proved their existence in 1928, they are used in popular science writing to explain that "complete chaos is impossible". However, determining these number exactly is a notoriously difficult problem, so typically the name of the game is to obtain good estimates instead.

Historically, the probabilistic method and random graph theory has been the leading source of progress in this area. Indeed, these areas were pioneered by Erdős himself to solve problems in Ramsey theory, and advances in the former were usually driven by questions in the latter. Despite significant development of the probabilistic method since its inception, many important problems in Ramsey theory remain open. In particular, we do not understand very well the rate of growth of off-diagonal Ramsey numbers, where we fix s and let t grow arbitrarily large. It is known that the dominant factor of this function is polynomial in t, but the correct degree has only been determined for $s = 3$.

In 2016, a new approach to this problem was suggested by Dhruv Mubayi and Jacques Verstraete, which implied that a mixture of algebraic and probabilistic techniques might provide an alternative to the probabilistic method. This was confirmed by Jacques Verstraete and myself when we are able to use this blueprint to determine the correct power of t in the asymptotics of the off-diagonal Ramsey number $r(4,t)$, thereby solving a 40-year-old problem and confirming a conjecture due to Erdős.